Proceedings of the American Academy of Arts and Sciences.

Vol. XLVI. No. 7. — October, 1910.

CONTRIBUTIONS FROM THE RESEARCH LABORATORY OF PHYSICAL CHEMISTRY OF THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY. — No. 58.

ON FOUR-DIMENSIONAL VECTOR ANALYSIS, AND ITS APPLICATION IN ELECTRICAL THEORY.

By Gilbert N. Lewis.

CONTRIBUTION FROM RESEARCH LABORATORY OF PHYSICAL CHEMISTRY OF THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY.—No. 58.

ON FOUR-DIMENSIONAL VECTOR ANALYSIS, AND ITS APPLICATION IN ELECTRICAL THEORY.

By Gilbert N. Lewis.

Received June 28, 1910.

The great generalization of Einstein, known as the principle of relativity, and its interpretation by Minkowski, have opened a new domain of natural science. The apparent artificiality and paradox of some of the consequences of the relativity theory disappear completely when with Minkowski we regard the science of kinematics as identical with the geometry of four-dimensional space.

Minkowski [1] and, following him, Abraham [2] have made an important beginning in the use of four-dimensional vector analysis. In general, however, Minkowski used for his more important deductions, not the vectorial method, but the matrix calculus of Cayley. This was undoubtedly due to the restricted and specialized character of our present vector analysis, for the vector method, permitting as it does a ready survey, and often a visualization of the results to which it leads, has shown its superiority over all other methods in several branches of physics, and there can be no doubt that it is also peculiarly well adapted to the solution of the new problems introduced by Minkowski.

I shall attempt to show in this paper what simple changes must be made in our present system of vector analysis to make it immediately adaptable to a space of higher dimensions. Only such changes will be made as are imperatively demanded by the nature of the problem, and these few changes will, I believe, recommend themselves, not only because of the increased generality of the resulting analysis, but because they restore many features of the original, and much neglected, system of Grassmann.[3]

---

[1] Göttingen, Nachricht., 1908, p. 53.
[2] Rendiconti di Palermo, **30**, 1 (1910).
[3] References to Grassmann will be to the edition of 1894, Teubner, Leipzig.

In the second section several of the most useful formulae of four-dimensional vector analysis will be presented, and the last section will be devoted to some applications of these formulae in electromagnetic theory and the theory of relativity.

The Vector Analysis of Three Dimensions.

The simplest type of quantity which is distinguished from others of its class by magnitude and direction is the familiar line-vector, a one-dimensional quantity which we shall call a vector of the first order, or in brief, a 1-vector.

Just as two parallel line-vectors of the same length are regarded as equal, so two parallel plane surfaces of the same area are also considered equal. A plane area [4] constitutes a vector of the second order, or a 2-vector.

In general in a space of n dimensions we may distinguish 0-vectors or scalars; 1-vectors, 2-vectors, 3-vectors, etc., up to the n-vectors, which, like the 0-vectors, have no direction and may therefore be called pseudo-scalars.

In three-dimensional space the only true vectors which exist are 1-vectors and 2-vectors. Moreover, every 2-vector determines uniquely the 1-vector normal to it. In common vector analysis the 2-vector is regarded as equivalent to and replaceable by its normal 1-vector of the same magnitude, and therefore this analysis deals solely with 1-vectors. This simplification has certain obvious advantages which, however, are for the most part superficial. In some cases moreover it leads to difficulties.[5] In any case it must be abandoned when we pass to space of higher dimensions, where a 2-vector no longer uniquely determines a 1-vector.

Our first departure, then, from common vector analysis will consist in distinguishing between vectors of different orders. A 1-vector will

---

[4] For simplicity we may deal only with the straight vectors (straight line, plane surface, etc.) since any curve terminating in two points may be regarded as equivalent to the straight line terminating in the same points, and a curved surface terminating in a *plane* closed curve, as equivalent to the plane area having the same boundary. Such a vector as a curved surface bounded by a closed curve which does not lie in a plane we shall not consider here.

[5] See, for example, the discussion of scalars and pseudo-scalars in Abraham-Föppl, Theorie der Elektrizität, p. 22–23. We shall have frequent occasion to cite this standard work, which contains an admirable presentation of the current system of vector analysis, as well as of electrical theory. References are to the edition of 1904 (Teubner, Leipzig).

be represented by a small letter in heavy type (e. g., $\mathbf{a}$, $\mathbf{s}$). A 2-vector will be represented by a capital letter in heavy type (e. g., $\mathbf{A}$, $\mathbf{S}$).

Let us consider a coördinate system of three perpendicular axes, x_1, x_2, x_3, and represent by $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, the three unit vectors in these three directions. If the lengths of the components of a 1-vector $\mathbf{a}$ on the three axes are a_1, a_2, a_3, then

$$\mathbf{a} = a_1\mathbf{k}_1 + a_2\mathbf{k}_2 + a_3\mathbf{k}_3. \tag{1}$$

In general the addition and subtraction of 1-vectors follow the law.[6]

$$\mathbf{a} \pm \mathbf{b} = (a_1 \pm b_1)\,\mathbf{k}_1 + (a_2 \pm b_2)\,\mathbf{k}_2 + (a_3 \pm b_3)\,\mathbf{k}_3. \tag{2}$$

Similarly we may project a surface vector, or 2-vector, upon the three coordinate planes determined by x_1, x_2; x_1, x_3; x_2, x_3. The unit 2-vectors in these planes we will denote by $\mathbf{k}_{12}$, $\mathbf{k}_{13}$, $\mathbf{k}_{23}$,[7] and the areas of the projections of a 2-vector $\mathbf{A}$ by A_{12}, A_{13}, A_{23}. Then,

$$A = A_{12}\mathbf{k}_{12} + A_{13}\mathbf{k}_{13} + A_{23}\mathbf{k}_{23} \tag{3}$$

$$A \pm B = (A_{12} \pm B_{12})k_{12} + (A_{13} \pm B_{13})k_{13} + (A_{23} \pm B_{23})k_{23}. \tag{4}$$

Further we may adopt the convention,

$$\mathbf{k}_{12} = -\,\mathbf{k}_{21}\,;\ \mathbf{k}_{13} = -\,\mathbf{k}_{31}\,;\ \mathbf{k}_{23} = -\,\mathbf{k}_{32} \tag{5}$$

which requires the further convention,

$$A_{12} = -\,A_{21}\,;\ A_{13} = -\,A_{31}\,;\ A_{23} = -\,A_{32}. \tag{6}$$

Just as $\mathbf{k}_1$ represents a vector of unit length, $\mathbf{k}_{12}$ one of unit area, $\mathbf{k}_{123}$ will represent unit volume. It is the unit 3-vector or, in three-dimensional space, the unit pseudo-scalar. We shall adopt the convention

$$\mathbf{k}_{123} = \mathbf{k}_{312} = \mathbf{k}_{231} = -\,\mathbf{k}_{132} = -\,\mathbf{k}_{213} = -\,\mathbf{k}_{321}. \tag{7}$$

Equations (5) and (7) may be expressed in the following general rule which we shall also adopt in space of higher dimensions: Interchanging any two adjacent subscripts of a unit vector changes the sign of the vector.

---

[6] The addition both of 1-vectors and 2-vectors is best defined geometrically. (See Grassmann, Ausdehnungslehre von 1844, p. 78.) The introduction of coördinate axes brings a foreign element into pure vector analysis, but on the other hand it will permit us to translate the vector equations more readily into Cartesian equations.

[7] In this case we depart from our rule that 2-vectors shall be represented by capital letters. On account of the subscripts there will be no confusion.

Multiplying any vector by a scalar, multiplies its magnitude by that scalar. Multiplication of a vector by a scalar follows the laws of association, commutation and distribution.

$$m\mathbf{a} = ma_1\mathbf{k}_1 + ma_2\mathbf{k}_2 + ma_3\mathbf{k}_3 \tag{8}$$

In multiplying one vector by another two kinds of product are to be distinguished, which, following Grassmann, we shall call the inner and outer products,[8] and define as follows.

The inner product follows the distributive and commutative laws. It is in general a vector and its order is the difference between the orders of the factors. (Thus the inner product of two 1-vectors is a 0-vector, or scalar, the inner product of a 1-vector and a 2-vector is a 1-vector.)

The outer product follows the distributive and associative laws. It is a vector of which the order is the sum of the orders of the factors. (Thus the outer product of two 1-vectors is a 2-vector, that of a 1-vector and a 2-vector is a 3-vector, which in three-dimensional space is a pseudo-scalar.)

The inner product of two vectors will be indicated merely by their juxtaposition, for example, $\mathbf{ab}$; $\mathbf{AB}$; $\mathbf{aA}$.

The outer product will be indicated by a cross [9] placed between two vectors, for example, $\mathbf{a}\times\mathbf{b}$; $\mathbf{A}\times\mathbf{B}$; $\mathbf{a}\times\mathbf{A}$.

Since both kinds of products follow the distributive law they may be completely defined by the rules governing the multiplication of the simple unit vectors. The rules for inner multiplication are as follows,

$$\left.\begin{array}{rl} \mathbf{k}_1\mathbf{k}_1 = 1\,; & \mathbf{k}_1\mathbf{k}_2 = \mathbf{k}_2\mathbf{k}_1 = 0 \\ \mathbf{k}_{12}\mathbf{k}_{12} = 1\,; & \mathbf{k}_{12}\mathbf{k}_{13} = \mathbf{k}_{13}\mathbf{k}_{12} = 0 \\ \mathbf{k}_{123}\mathbf{k}_{123} = 1\,; & \\ \mathbf{k}_2\mathbf{k}_{12} = \mathbf{k}_{12}\mathbf{k}_2 = \mathbf{k}_1\,; & \mathbf{k}_1\mathbf{k}_{23} = \mathbf{k}_{23}\mathbf{k}_1 = 0 \\ \mathbf{k}_3\mathbf{k}_{123} = \mathbf{k}_{123}\mathbf{k}_3 = \mathbf{k}_{12}\,; & \\ \mathbf{k}_{23}\mathbf{k}_{123} = \mathbf{k}_{123}\mathbf{k}_{23} = \mathbf{k}_1\,; & \end{array}\right\}. \tag{9}$$

These statements may be generalized, and the following rules will hold also for unit vectors mutually perpendicular, in space of any dimensions :

[8] The terms scalar product and vector product would obviously be misnomers in the present system.

[9] This symbol for the outer product is used by Gibbs and his followers (see Gibbs, Collected Papers; Wilson-Gibbs, Vector Analysis; Coffin, Vector Analysis), and has several advantages over the more awkward square brackets [a, b] frequently used to express the outer product. The brackets were used by Grassmann, but had a far more general significance than the product defined above. (Ausdehnungslehre von 1862, p. 28.)

If each factor has a subscript which the other has not, the inner product is zero.

The inner product of two identical unit vectors is equal to unity.

In the remaining case, one factor having a higher order than the other, the subscripts of the former should be transposed until those subscripts occurring at the right are the same and in the same sequence as in the factor of lower order. These common subscripts are then cancelled and a unit vector with the remaining subscripts, in the sequence in which they stand, forms the inner product. Thus for example,

$$\mathbf{k}_{13}\mathbf{k}_{123} = -\,\mathbf{k}_{13}\mathbf{k}_{213} = -\,\mathbf{k}_2.$$

From these rules we obtain immediately the equations,

$$\mathbf{ab} = a_1b_1 + a_2b_2 + a_3b_3. \qquad (10)$$

$$\mathbf{aa} = \mathbf{a}^2 = a_1{}^2 + a_2{}^2 + a_3{}^2. \qquad (11)$$

$$\mathbf{AB} = A_1B_1 + A_2B_2 + A_3B_3. \qquad (12)$$

These products are scalars. On the other hand the product $\mathbf{aA}$ is a 1-vector lying in the plane $\mathbf{A}$ and perpendicular to the projection of $\mathbf{a}$ upon $\mathbf{A}$, namely,

$$\mathbf{aA} = \mathbf{Aa} = (A_{12}a_2 + A_{13}a_3)\,\mathbf{k}_1 + (A_{21}a_1 + A_{23}a_3)\,\mathbf{k}_2 + (A_{31}a_1 + A_{32}a_2)\,\mathbf{k}_3. \qquad (13)$$

Finally, the product of a pseudo-scalar and a 1-vector is the perpendicular 2-vector, that of a pseudo-scalar and a 2-vector is the perpendicular 1-vector, the product in each case having a magnitude equal to the product of the magnitudes of the factors.

The rules for outer multiplication may likewise be stated by stating the rules for the unit vectors.

$$\begin{gathered}\mathbf{k}_1\times\mathbf{k}_2 = \mathbf{k}_{12}\,;\ \mathbf{k}_1\times\mathbf{k}_{23} = \mathbf{k}_{12}\times\mathbf{k}_3 = \mathbf{k}_{123},\\ \mathbf{k}_1\times\mathbf{k}_1 = 0\,;\ \mathbf{k}_1\times\mathbf{k}_{12} = 0\,;\ \mathbf{k}_{12}\times\mathbf{k}_{12} = 0\,;\ \mathbf{k}_{12}\times\mathbf{k}_{13} = 0\,;\\ \mathbf{k}_1\times\mathbf{k}_{123} = 0\,;\ \mathbf{k}_{12}\times\mathbf{k}_{123} = 0\,;\ \mathbf{k}_{123}\times\mathbf{k}_{123} = 0.\end{gathered} \qquad (14)$$

These statements may be generalized and the following rules will hold also for unit, mutually perpendicular, vectors in space of any dimensions :

If two unit vectors possess any subscript in common, their outer product is zero.

In all other cases the outer product is a unit vector having all the subscripts of both factors, in the sequence in which they occur in the factors.

From these rules[10] we obtain the equations,

$$\mathbf{a}\times\mathbf{a} = 0. \tag{15}$$

$$\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a} = (a_1b_2 - a_2b_1)\,\mathbf{k}_{12} + (a_1b_3 - a_3b_1)\,\mathbf{k}_{13} + (a_2b_3 - a_3b_2)\,\mathbf{k}_{23}. \tag{16}$$

$$\mathbf{a}\times\mathbf{A} = \mathbf{A}\times\mathbf{a} = (a_1A_{23} + a_2A_{31} + a_3A_{12})\,\mathbf{k}_{123}. \tag{17}$$

When the 2-vector in (17) is expressed as a product of two 1-vectors, $\mathbf{b}\times\mathbf{c}$, that equation becomes

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\times\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\,\mathbf{k}_{123}. \tag{18}$$

$$\text{and } \mathbf{a}\times(\mathbf{a}\times\mathbf{b}) = 0. \tag{18a}$$

We thus see that $\mathbf{a}\times\mathbf{b}$ represents the parallelogram determined by **a** and **b**, and $\mathbf{a}\times\mathbf{b}\times\mathbf{c}$ the parallelopiped determined by **a**, **b** and **c**.

It is important at this point to rewrite equation (13) using $\mathbf{b}\times\mathbf{c}$ in place of **A**; expanding and rearranging the terms gives

$$\mathbf{a}\,(\mathbf{b}\times\mathbf{c}) = \mathbf{ab}\times\mathbf{c}\,^{11} = (\mathbf{ac})\,\mathbf{b} - (\mathbf{ab})\,\mathbf{c}. \tag{19}$$

These equations (18) and (19) deserve especial attention, for they show the only essential difference between our system and the common system of vector analysis. The two systems give the same result for the outer multiplication of two 1-vectors and for the inner product of two 1-vectors or two 2-vectors. But the meanings of the outer and inner products of **a** and $\mathbf{b}\times\mathbf{c}$ are just reversed in the two systems.

Finally we find from our rules for unit vectors the outer product of two 2-vectors,

$$\mathbf{A}\times\mathbf{B} = 0. \tag{20}$$

By our general principle the outer product must in this case have the order 2 + 2, and a 4-vector cannot exist in three-dimensional space.[12]

---

[10] The rules here given are somewhat redundant. For example, the distributive law, and $\mathbf{a}\times\mathbf{a} = 0$, alone suffice to prove $\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}$, for

$$(a + b) \times (a + b) = 0 = \mathbf{a}\times\mathbf{a} + \mathbf{b}\times\mathbf{b} + \mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{a} = \mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{a}.$$

See Grassmann, Ausdehnungslehre von 1844, p. 87.

[11] The parentheses may be removed simply because $(\mathbf{ab}) \times \mathbf{c}$ has no meaning.

[12] In ordinary vector analysis a meaning is given to the outer product of $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$. It represents a vector determined by the line of intersection of the surface $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$. We have seen that in n-dimensional space an n-vector has some properties of a scalar or vector of the order $n-n$. So we may modify our rules of multiplication so that the product of a p-vector and a

The inner product of $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$ may be obtained from the preceding equations, and is the same as in ordinary vector analysis,

$$(\mathbf{a}\times\mathbf{b})\,(\mathbf{c}\times\mathbf{d}) = (\mathbf{ac})\,(\mathbf{bd}) - (\mathbf{bc})\,(\mathbf{ad}). \tag{21}$$

The differential operator ∇ ("del") we may define in the usual way,[13] namely,

$$\nabla = \mathbf{k}_1 \frac{\partial}{\partial x_1} + \mathbf{k}_2 \frac{\partial}{\partial x_2} + \mathbf{k}_3 \frac{\partial}{\partial x_3}. \tag{22}$$

Since the scalar operator $\frac{\partial}{\partial x}$, when applied to a single variable, can be treated as an algebraic quantity, the operator ∇ may be treated formally as a 1-vector, and we may derive a number of important equations by substituting ∇ for $\mathbf{a}$ or $\mathbf{b}$ in the preceding equations. Thus from (8) we obtain from the scalar ϕ the function known as gradient of ϕ

$$\nabla\phi = \mathbf{k}_1 \frac{\partial\phi}{\partial x_1} + \mathbf{k}_2 \frac{\partial\phi}{\partial x_2} + \mathbf{k}_3 \frac{\partial\phi}{\partial x_3}. \tag{23}$$

Combined with a 1-vector by inner and outer multiplication we obtain by equations (10) and (16), the functions known as the divergence and curl, respectively.

$$\nabla\mathbf{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}. \tag{24}$$

$$\nabla\times\mathbf{a} = \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathbf{k}_{12} + \left(\frac{\partial a_3}{\partial x_1} - \frac{\partial a_1}{\partial x_3}\right)\mathbf{k}_{13} + \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\mathbf{k}_{23}. \tag{25}$$

Evidently $\nabla\phi$ is a 1-vector, $\nabla\mathbf{a}$ a scalar, and $\nabla\times\mathbf{a}$ a 2-vector.

By equations (13) and (17) we may write expressions for $\nabla\mathbf{A}$ (a 1-vector), and $\nabla\times\mathbf{A}$ (a 3-vector, or pseudo scalar).

---

q-vector, when $p + q > n$, is a vector of the order $p + q - n$. Such a product, which Grassmann calls "regressive," is formed according to a new set of rules and may best be regarded as a new type of product entirely distinct from the regular or "progressive" outer product. It is possible in the system here described to avoid the introduction of this new kind of product. Thus

$$((\mathbf{a}\times\mathbf{b})\,\mathbf{k}_{123}) \times (\,(\mathbf{c}\times\mathbf{d}\;\mathbf{k}_{123}) = \mathbf{e}\mathbf{k}_{123}),$$

where $\mathbf{e}$ is the 1-vector obtained in the ordinary vector analysis as the outer product of $(\mathbf{a}\times\mathbf{b})$ and $(\mathbf{c}\times\mathbf{d})$.

[13] Like other vector quantities and operators ∇ may be simply defined without reference to coördinates. See, for example, Wilson, Bull. Amer. Math. Soc. (2) **16**, 415 (1910).

$$\nabla \mathbf{A} = \left(\frac{\partial A_{12}}{\partial x_2} + \frac{\partial A_{13}}{\partial x_3}\right)\mathbf{k}_1 + \left(\frac{\partial A_{21}}{\partial x_1} + \frac{\partial A_{23}}{\partial x_3}\right)\mathbf{k}_2 + \left(\frac{\partial A_{31}}{\partial x_1} + \frac{\partial A_{32}}{\partial x_2}\right)\mathbf{k}_3. \tag{26}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_{23}}{\partial x_1} + \frac{\partial A_{31}}{\partial x_2} + \frac{\partial A_{12}}{\partial x_3}\right)\mathbf{k}_{123}. \tag{27}$$

When the quantity operated upon by ∇ contains two or more variables it may be expanded in terms of its components and these scalar quantities may then be differentiated in the ordinary way. We thus obtain such equations as the following :

$$\nabla (\phi \mathbf{a}) = \phi (\nabla \mathbf{a}) + \mathbf{a} (\nabla \phi), \tag{28}$$

$$\nabla \times (\phi \mathbf{a}) = \phi (\nabla \times \mathbf{a}) + (\nabla \phi) \times \mathbf{a}, \tag{29}$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \times (\nabla \times \mathbf{a}) - \mathbf{a} \times (\nabla \times \mathbf{b}). \tag{30}$$

By the above rules new operators may be formed from ∇ such as $\mathbf{a}\nabla$, $\mathbf{A}\nabla$, and $\nabla\nabla$ or ∇^2 which may be applied to any scalar or vector. The last is the well-known Laplacian operator and may obviously be expanded by equation (11),

$$\nabla^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2}. \tag{31}$$

Other operations involving ∇ twice are $\nabla(\nabla \mathbf{a})$ and $\nabla(\nabla \times \mathbf{a})$ or $\nabla\nabla\times\mathbf{a}$.

These quantities are connected by an important equation which we obtain by expanding according to (13), (23) and (10), namely

$$\nabla \nabla \times \mathbf{a} = \nabla (\nabla \mathbf{a}) - \nabla^2 \mathbf{a}. \tag{32}$$

Finally we have from (18a) and (15) the important identities,

$$\nabla \times (\nabla \times \mathbf{a}) = 0. \tag{33}$$

$$\nabla \times (\nabla \phi) = 0. \tag{34}$$

Equations (32), (33) and (34) are evidently equivalent to the familiar equations :[14]

$$\begin{aligned} &\text{curl}^2\, \mathbf{a} = \text{grad}\,(\text{div}\, \mathbf{a}) - \nabla^2 \mathbf{a}, \\ &\text{div curl}\, \mathbf{a} = 0. \\ &\text{curl grad}\, \phi = 0. \end{aligned}$$

[14] Abraham-Föppl, 1, equations 95, 94, 91 a.

Here as elsewhere our equations differ from those in common use whenever the product of a 1-vector and a 2-vector is concerned.

The Vector Analysis of Four Dimensions.

The revised system of three-dimensional vector analysis has been elaborated somewhat fully in the preceding section, since the methods there adopted may be used without any modification in developing the vector analysis of space of higher dimensions.

Let us consider a four-dimensional space in which any two points uniquely determine a straight line, any three points not in a line uniquely determine a plane, and any four points not in a plane uniquely determine a straight or Euclidean 3-space. This may be called a Euclidean four-dimensional space.

In such a space let us construct four mutually perpendicular coordinate axes, x_1, x_2, x_3, x_4. The 1-vectors of unit length in these four directions we may call $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4$. Each pair of axes determines a plane, thus forming six coördinate planes. The 2-vectors of unit area parallel to these planes we may call $\mathbf{k}_{12}, \mathbf{k}_{13}, \mathbf{k}_{14}, \mathbf{k}_{23}, \mathbf{k}_{24}, \mathbf{k}_{34}$. These six planes are mutually perpendicular. Moreover the plane $\mathbf{k}_{12}$ is *completely* perpendicular to the plane $\mathbf{k}_{34}$, in the sense that every line in $\mathbf{k}_{12}$ is perpendicular to every line in $\mathbf{k}_{34}$. The same is true of the pairs $\mathbf{k}_{23}, \mathbf{k}_{14}$, and $\mathbf{k}_{13}, \mathbf{k}_{24}$.

Each set of three axes determines a straight 3-space and the four coördinate 3-spaces thus determined may be represented by the unit 3-vectors $\mathbf{k}_{123}, \mathbf{k}_{124}, \mathbf{k}_{134}, \mathbf{k}_{234}$. Finally all four axes together determine the unit 4-vector or pseudo-scalar, $\mathbf{k}_{1234}$.

A 1-vector may be represented as the sum of its projections on the four axes,

$$\mathbf{a} = a_1\mathbf{k}_1 + a_2\mathbf{k}_2 + a_3\mathbf{k}_3 + a_4\mathbf{k}_4. \tag{35}$$

A 2-vector may be represented as the sum of its projections on the six coördinate planes.

$$\mathbf{A} = A_{12}\mathbf{k}_{12} + A_{13}\mathbf{k}_{13} + A_{14}\mathbf{k}_{14} + A_{23}\mathbf{k}_{23} + A_{24}\mathbf{k}_{24} + A_{34}\mathbf{k}_{34}. \tag{36}$$

A 3-vector may likewise be represented as the sum of its four projections on the coördinate 3-spaces.[15]

The addition and subtraction of vectors follow the same rules as in the case of three dimensions (equations 2 and 4). Moreover both

[15] We shall not use the 3-vectors often enough in this paper to justify the introduction of a special symbol for them.

forms of multiplication are completely defined by the distributive law, and by the rules already given for the transposition of subscripts, and for inner and outer multiplication among the unit vectors. We may therefore write at once a large number of equations, of which some of the more important are the following,

$$\mathbf{A}\times\mathbf{B} = \mathbf{B}\times\mathbf{A} = (A_{12}B_{34} + A_{13}B_{24} + A_{14}B_{23} + A_{23}B_{14} + A_{24}B_{13} + A_{34}B_{12})\mathbf{k}_{1234}. \quad (37)$$

$$\mathbf{a}\times\mathbf{A} = \mathbf{A}\times\mathbf{a} = (a_1A_{23} + a_2A_{31} + a_3A_{12})\mathbf{k}_{123} + (a_1A_{24} + a_2A_{41} + a_4A_{12})\mathbf{k}_{124} + (a_1A_{34} + a_3A_{41} + a_4A_{13})\mathbf{k}_{134} + (a_2A_{34} + a_3A_{42} + a_4A_{23})\mathbf{k}_{234}. \quad (38)$$

$$\mathbf{a}\times\mathbf{b} = (a_1b_2 - a_2b_1)\mathbf{k}_{12} + (a_1b_3 - a_3b_1)\mathbf{k}_{13} + \,.\,. \quad (39)$$

$$\mathbf{a}\times\mathbf{a} = 0. \quad (40)$$

Here also $\mathbf{a}\times\mathbf{b}$ evidently represents the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$, so $\mathbf{a}\times\mathbf{b}\times\mathbf{c}$ will represent a parallelopiped, and $\mathbf{a}\times\mathbf{b}\times\mathbf{c}\times\mathbf{d}$ a parallel four-dimensional figure. It is very important to observe that all of our four-dimensional vector equations may be given simple geometrical definitions, and retain complete validity whatever set of coördinate axes be arbitrarily chosen.

Some of the more important inner products are the following :

$$\mathbf{ab} = \mathbf{ba} = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4. \quad (41)$$

$$\mathbf{AB} = \mathbf{BA} = A_{12}B_{12} + A_{13}B_{13} + A_{14}B_{14} + A_{23}B_{23} + A_{24}B_{24} + A_{34}B_{34}. \quad (42)$$

$$\mathbf{aA} = \mathbf{Aa} = (A_{12}a_2 + A_{13}a_3 + A_{14}a_4)\mathbf{k}_1 + (A_{21}a_1 + A_{23}a_3 + A_{24}a_4)\mathbf{k}_2 + \,.\,.\,. \quad (43)$$

$$(\mathbf{a}\times\mathbf{b})\,(\mathbf{c}\times\mathbf{d}) = (\mathbf{ac})\,(\mathbf{bd}) - (\mathbf{bc})\,(\mathbf{ad}). \quad (44)$$

$$\mathbf{a}\,(\mathbf{b}\times\mathbf{c}) = (\mathbf{ac})\,\mathbf{b} - (\mathbf{ab})\,\mathbf{c}. \quad (45)$$

This is a 1-vector lying in the plane $\mathbf{b}\times\mathbf{c}$ and perpendicular to the projection of $\mathbf{a}$ thereon. So $\mathbf{a}\,(\mathbf{b}\times\mathbf{c}\times\mathbf{d})$ is a 2-vector lying in the 3-space $\mathbf{b}\times\mathbf{c}\times\mathbf{d}$ and perpendicular to the projection of $\mathbf{a}$ on that 3-space; $(\mathbf{a}\times\mathbf{b})\,(\mathbf{c}\times\mathbf{d}\times\mathbf{e})$ is a 1-vector in the 3-space $\mathbf{c}\times\mathbf{d}\times\mathbf{e}$ and perpendicular to the projection of $\mathbf{a}\times\mathbf{b}$ thereon.

The inner product of any vector with unit pseudo-scalar, $\mathbf{k}_{1234}$, is another vector of the same magnitude which may be called its *complement*. The complement of a scalar is a pseudo-scalar, and vice versa. The complement of a 1-vector is a 3-vector normal to it, and vice versa.

The complement of a 2-vector is the completely perpendicular 2-vector. In the vector analysis at present in use it is customary to identify a vector with its complement, and this is also done by Abraham[16] in the paper in which he makes use of four-dimensional vector analysis. In our present analysis there is no advantage to be gained by this step, which may cause much confusion.

As in the case of three dimensions we may define a differential operator, having the form of a 1-vector, as follows:

$$\diamond = \mathbf{k}_1 \frac{\partial}{\partial x_1} + \mathbf{k}_2 \frac{\partial}{\partial x_2} + \mathbf{k}_3 \frac{\partial}{\partial x_3} + \mathbf{k}_4 \frac{\partial}{\partial x_4}. \tag{46}$$

This operator[17] $\diamond$ (read "quad") may be treated like a simple vector under the same conditions as in the case of ∇. We thus obtain a number of important equations such as the following.

$$\diamond\phi = \mathbf{k}_1 \frac{\partial\phi}{\partial x_1} + \mathbf{k}_2 \frac{\partial\phi}{\partial x_2} + \mathbf{k}_3 \frac{\partial\phi}{\partial x_3} + \mathbf{k}_4 \frac{\partial\phi}{\partial x_4}. \tag{47}$$

$$\diamond \cdot \mathbf{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} + \frac{\partial a_4}{\partial x_4}. \tag{48}$$

$$\diamond\times\mathbf{a} = \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathbf{k}_{12} + \left(\frac{\partial a_3}{\partial x_1} - \frac{\partial a_1}{\partial x_3}\right)\mathbf{k}_{13} + \ldots \tag{49}$$

These three expressions correspond to gradient, divergence and curl in three-dimensional analysis. We may also apply $\diamond$ to vectors of higher orders, for example, by (43) and (38),

$$\diamond \cdot \mathbf{A} = \left(\frac{\partial A_{12}}{\partial x_2} + \frac{\partial A_{13}}{\partial x_3} + \frac{\partial A_{14}}{\partial x_4}\right)\mathbf{k}_1 + \left(\frac{\partial A_{21}}{\partial x_1} + \frac{\partial A_{23}}{\partial x_3} + \frac{\partial A_{24}}{\partial x_4}\right)\mathbf{k}_2 + \ldots \tag{50}$$

$$\diamond\times\mathbf{A} = \left(\frac{\partial A_{23}}{\partial x_1} + \frac{\partial A_{31}}{\partial x_2} + \frac{\partial A_{12}}{\partial x_3}\right)\mathbf{k}_{123} + \ldots \tag{51}$$

We may form other operators like, $\mathbf{A}\diamond$, a 1-vector operator, and the scalar operators $\mathbf{a}\cdot\diamond$, and $\diamond^2$. The last is the very important operator which Lorentz in a special case calls the d'Alembertian operator,

$$\diamond^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2} + \frac{\partial^2}{\partial x_4{}^2}. \tag{52}$$

---

[16] Abraham (loc. cit.).

[17] The operator $\diamond$ has the same scalar components as the operator *lor* used by Minkowski.

Any of these operators may be applied to a scalar or to a vector of any order.

Two other important operations are connected with $\Diamond^2$ by the formula analogous to (32):

$$\Diamond(\Diamond\times\mathbf{a}) = \Diamond\Diamond\times\mathbf{a} = \Diamond(\Diamond\,\mathbf{a}) - \Diamond^2\mathbf{a}. \qquad (53)$$

And we have here also two important identities

$$\Diamond\times(\Diamond\times\mathbf{a}) = 0, \qquad (54)$$

$$\Diamond\times(\Diamond\phi) = 0. \qquad (55)$$

In the same way that we obtained equations (28), (29), (30), we find

$$\Diamond(\phi\mathbf{a}) = \phi(\Diamond\mathbf{a}) + \mathbf{a}\,(\Diamond\phi), \qquad (56)$$

$$\Diamond\times(\phi\mathbf{a}) = \phi\,(\Diamond\times\mathbf{a}) + (\Diamond\phi)\times\mathbf{a}, \qquad (57)$$

$$\Diamond\times(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\times(\Diamond\times\mathbf{a}) - \mathbf{a}\times(\Diamond\times\mathbf{b}). \qquad (58)$$

These equations will suffice to illustrate how readily the generalized vector analysis of the preceding section may be applied in a space of any dimensions.

Some Applications of Four-Dimensional Vector Analysis in the Theory of Electricity.

The principle of relativity as interpreted by Minkowski can be summed up in the statement[18] that a Euclidean four-dimensional space is determined by the coördinates, x, y, z, and ict, where i is the unit of imaginaries, $\sqrt{-1}$; and c is the velocity of light. The whole science of kinematics is merely the geometry of this four-dimensional space. As Minkowski himself has shown, there is no domain in which this new conception is more fruitful than in the science of electricity and magnetism.

Let us consider a system composed of electric charges moving in free space. The density of charge at any point we may call $\frac{\varrho}{c}$ in electromagnetic units, and if we call the velocity of the charge $\mathbf{v}$, then $\frac{\varrho}{c}\mathbf{v}$ represents the current density at a point. This 1-vector $\frac{\varrho}{c}\mathbf{v}$ lies wholly in the 3-space x, y, z.

[18] This statement is subject to certain restrictions that we will not discuss.

Following Minkowski we may define a 1-vector, $\mathbf{q}$, in the space x, y, z, ict (or x_1, x_2, x_3, x_4) of which the projection on the 3-space is $\frac{\varrho}{c}\mathbf{v}$ and the scalar component along the x_4 (or ict) axis is $i\varrho$, by the equation,

$$\mathbf{q} = \frac{\varrho}{c}\mathbf{v} + i\,\varrho\,\mathbf{k}_4, \tag{59}$$

or

$$\mathbf{q} = \frac{\varrho}{c}v_1\mathbf{k}_1 + \frac{\varrho}{c}v_2\mathbf{k}_2 + \frac{\varrho}{c}v_3\mathbf{k}_3 + i\,\varrho\,\mathbf{k}_4. \tag{60}$$

Furthermore, from the electrical force $\mathbf{e}$, and the magnetic force $\mathbf{h}$, we shall find it convenient to define two new 2-vectors, $\mathbf{E}$ and $\mathbf{H}$, by the equations,[19]

$$\mathbf{E} = -\,i\mathbf{e}\times\mathbf{k}_4, \tag{61}$$

and

$$\mathbf{H} = \mathbf{h}\mathbf{k}_{123}. \tag{62}$$

$\mathbf{H}$ is the 2-vector complementary to $\mathbf{h}$ in the 3-space x, y, z. From these definitions we have

$$H_{12} = h_3\,;\; H_{23} = h_1\,;\; H_{31} = h_2, \tag{63}$$

$$E_{14} = -\,ie_1\,;\; E_{24} = -\,ie_2\,;\; E_{34} = -\,ie_3. \tag{64}$$

From $\mathbf{H}$ and $\mathbf{e}$ we may define[20] the vector potential $\mathbf{a}$ and the scalar potential ϕ by the familiar equations,

$$\mathbf{H} = \nabla\times\mathbf{a}, \tag{65}$$

$$-\,\mathbf{e} = \nabla\phi + \frac{1}{c}\frac{\partial\mathbf{a}}{\partial t}. \tag{66}$$

Finally we shall define a new 1-vector m by the equation,

$$\mathbf{m} = \mathbf{a} + i\phi\mathbf{k}_4, \tag{67}$$

or

$$\mathbf{m} = a_1\mathbf{k}_1 + a_2\mathbf{k}_2 + a_3\mathbf{k}_3 + i\phi\mathbf{k}_4. \tag{68}$$

Thus m is a vector of which the projection on the 3-space x, y, z is the vector potential, and of which the scalar component in the ict direction is the scalar potential multiplied by $\sqrt{-1}$.

---

[19] Compare in this connection the discussion of $\mathbf{e}$ as a "polar" vector, $\mathbf{h}$ as an "axial" vector, in Abraham-Föppl (1, p. 243).

[20] This definition is evidently not complete, since $\mathbf{a}$ and ϕ are derived from $\mathbf{H}$ and $\mathbf{e}$ by a process of integration. We shall return to this point.

We shall call $\mathbf{m}$ the *fundamental electromagnetic 1-vector*, and show that the four important field equations of Maxwell and Lorentz, as well as other well-known equations, are all contained in the strikingly simple formula,

$$\diamond\diamond\times\mathbf{m} = \mathbf{q}. \tag{69}$$

In addition to this equation which states the experimental facts, we have from (54) the important identity

$$\diamond\times\diamond\times\mathbf{m} = 0. \tag{70}$$

The quantity $\diamond\times\mathbf{m}$, which might be called the four-dimensional curl of $\mathbf{m}$, is the fundamental electromagnetic 2-vector.[21] We will give it the symbol $\mathbf{M}$,

$$\diamond\times\mathbf{m} = \mathbf{M}. \tag{71}$$

Expanding $\diamond\times\mathbf{m}$ by equation (49) gives

$$\mathbf{M} = \mathbf{k}_{12}\left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) + \mathbf{k}_{13}\left(\frac{\partial a_3}{\partial x_1} - \frac{\partial a_1}{\partial x_3}\right) + \mathbf{k}_{23}\left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)$$
$$+ \mathbf{k}_{14}\left(\frac{\partial i\phi}{\partial x_1} - \frac{\partial a_1}{\partial x_4}\right) + \mathbf{k}_{24}\left(\frac{\partial i\phi}{\partial x_2} - \frac{\partial a_2}{\partial x_4}\right) + \mathbf{k}_{34}\left(\frac{\partial i\phi}{\partial x_3} - \frac{\partial a_3}{\partial x_4}\right). \tag{72}$$

The first three terms are evidently equal to the curl of $\mathbf{a}$; the last three may be put in the form

$$\left(i\mathbf{k}_1\frac{\partial\phi}{\partial x_1} + i\mathbf{k}_2\frac{\partial\phi}{\partial x_2} + i\mathbf{k}_3\frac{\partial\phi}{\partial x_3} - \frac{\partial a}{\partial x_4}\right)\times\mathbf{k}_4;$$

and further collecting terms, and writing ict for x_4, gives

$$i\left(\nabla\phi + \frac{1}{c}\frac{\partial\mathbf{a}}{\partial t}\right)\times\mathbf{k}_4.$$

Hence,

$$\mathbf{M} = \nabla\times\mathbf{a} + i\left(\nabla\phi + \frac{1}{c}\frac{\partial\mathbf{a}}{\partial t}\right)\times\mathbf{k}_4, \tag{73}$$

and thus by equations (61), (65) and (66)

$$\mathbf{M} = \mathbf{H} + \mathbf{E}. \tag{74}$$

This equation gives a better idea of the physical significance of the 2-vector $\diamond\times\mathbf{m}$, or $\mathbf{M}$. $\mathbf{H}$ is a 2-vector lying wholly in the x, y, z 3-space. $\mathbf{E}$ is a 2-vector perpendicular to the x, y, z 3-space in a plane determined by the 1-vectors $\mathbf{e}$ and $\mathbf{k}_4$. We may therefore write,

[21] This is the equivalent of the f or F of Minkowski.

$$\mathbf{M} = M_{12}\mathbf{k}_{12} + M_{13}\mathbf{k}_{13} + M_{23}\mathbf{k}_{23} + M_{14}\mathbf{k}_{14} + M_{24}\mathbf{k}_{24} + M_{34}\mathbf{k}_{34},$$
$$= H_{12}\mathbf{k}_{12} + H_{13}\mathbf{k}_{13} + H_{23}\mathbf{k}_{23} + E_{14}\mathbf{k}_{14} + E_{24}\mathbf{k}_{24} + E_{34}\mathbf{k}_{34}. \tag{75}$$

By our fundamental equation (69) we have

$$\Diamond\mathbf{M} = \mathbf{q}.$$

Expanding this equation by (50) with the aid of (75) gives four equations

$$\begin{aligned}
\left(\frac{\partial H_{12}}{\partial x_2} + \frac{\partial H_{13}}{\partial x_3} + \frac{\partial E_{14}}{\partial x_4}\right)\mathbf{k}_1 &= \frac{\rho}{c}\, v_1\mathbf{k}_1,\\
\left(\frac{\partial H_{21}}{\partial x_1} + \frac{\partial H_{23}}{\partial x_3} + \frac{\partial E_{24}}{\partial x_4}\right)\mathbf{k}_2 &= \frac{\rho}{c}\, v_2\mathbf{k}_2,\\
\left(\frac{\partial H_{31}}{\partial x_1} + \frac{\partial H_{32}}{\partial x_2} + \frac{\partial E_{34}}{\partial x_4}\right)\mathbf{k}_3 &= \frac{\rho}{c}\, v_3\mathbf{k}_3,\\
\left(\frac{\partial E_{41}}{\partial x_1} + \frac{\partial E_{42}}{\partial x_2} + \frac{\partial E_{43}}{\partial x_3}\right)\mathbf{k}_4 &= \rho\, i\mathbf{k}_4.
\end{aligned} \tag{76}$$

Collecting the first three equations into one, with the aid of (25), (63) and (64), and writing ict for x_4 gives

$$\nabla\times\mathbf{h} - \frac{1}{c}\frac{\partial\mathbf{e}}{\partial t} = \frac{\rho}{c}\,\mathbf{v}, \tag{77}$$

and the last equation by (24) and (64) changing E_{41} to $-E_{14}$, etc., gives

$$\nabla\,\mathbf{e} = \rho. \tag{78}$$

By a similar expansion of equation (70)

$$\Diamond\times\mathbf{M} = 0,$$

we find with the aid of (49)

$$\nabla\times\mathbf{e} + \frac{1}{c}\frac{\partial\mathbf{h}}{\partial t} = 0, \tag{79}$$

and

$$\nabla\,\mathbf{h} = 0. \tag{80}$$

Equations 77–80, in the more familiar notation, are the well-known equations,

$$\text{curl}\ \mathbf{h} - \frac{1}{c}\frac{\partial\mathbf{e}}{\partial t} = \frac{\rho}{c}\,\mathbf{v}, \tag{α}$$

$$\text{div}\ \mathbf{e} = \rho, \tag{β}$$

$$\text{curl}\ \mathbf{e} + \frac{1}{c}\frac{\partial\mathbf{h}}{\partial t} = 0, \tag{γ}$$

$$\text{div}\ \mathbf{h} = 0. \tag{δ}$$

Let us return to the discussion of the complete definition of the vector $\mathbf{m}$. All that we have hitherto said of this vector is comprised in the statement,

$$\Diamond \times \mathbf{m} = \mathbf{E} + \mathbf{H}.$$

It is evident that this equation does not completely define $\mathbf{m}$, for in general if $\mathbf{m}'$ is a vector satisfying the equation

$$\Diamond \times \mathbf{m}' = \mathbf{E} + \mathbf{H},$$

we may superpose upon the field of the vector $\mathbf{m}'$ the field of another vector $\mathbf{m}''$ for which

$$\Diamond \times \mathbf{m}'' = 0.$$

Then if $\mathbf{m} = \mathbf{m}' + \mathbf{m}''$, we also have

$$\Diamond \times \mathbf{m} = \mathbf{E} + \mathbf{H}.$$

Suppose now [22] that $\mathbf{m}''$ be so chosen that at every point in the field

$$\Diamond \mathbf{m}'' = - \Diamond \mathbf{m}'.$$

Then $\mathbf{m}$ satisfies the two equations,

$$\Diamond \times \mathbf{m} = \mathbf{E} + \mathbf{H},$$
$$\Diamond \mathbf{m} = 0. \tag{81}$$

We may, therefore, without in any way modifying what has preceded, complete the definition of $\mathbf{m}$ by the equation (81). This equation combined with (67) gives the well-known expression,

$$\nabla \mathbf{a} + \frac{\partial i\phi}{\partial x_4} = 0, \tag{82}$$

or

$$\operatorname{div} \mathbf{a} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0.$$ [23]

Now by equations (53) and (69),

$$\Diamond \times \mathbf{m} = \Diamond\,(\Diamond \mathbf{m}) - \Diamond^2 \mathbf{m} = \mathbf{q},$$

or by (81)

$$\Diamond^2 \mathbf{m} = - \mathbf{q}. \tag{83}$$

---

[22] This is not offered as a rigorous proof, for we have assumed that a field can be chosen with pre-determined values of $\Diamond \mathbf{m}''$ and $\Diamond \times \mathbf{m}''$.

[23] Abraham-Föppl, II, equation (30).

This is another simple form of our fundamental equation. Substituting for **m** by (67) gives the important equations[24]

$$\nabla^2\mathbf{a} - \frac{1}{c^2}\frac{\partial^2\mathbf{a}}{\partial t^2} = -\frac{\rho}{c}\mathbf{v}, \tag{84}$$

$$\nabla^2\phi - \frac{1}{c}\frac{\partial^2\phi}{\partial t^2} = -\rho. \tag{85}$$

Let us emphasize once more that all the equations of this section are mere definitions, or purely mathematical deductions, with the sole exception of the one equation which embodies the experimental facts, namely,

$$\Diamond\Diamond\times\mathbf{m} = \mathbf{q}.$$

In conclusion let us consider what is meant by the rotation of the axes in this four-dimensional space. The theory of relativity, as here employed, is equivalent to the statement that our four-dimensional vector equations are invariant in any orthogonal transformation of the axes x, y, z, ict.

The axis ict is characterized by the equation $\frac{\partial x}{\partial t} = \frac{\partial y}{\partial t} = \frac{\partial z}{\partial t} = 0$ and may be regarded as the four-dimensional locus ("Weltlinie") of a point at rest. A straight line, making a small angle with this axis in the plane passing through x and ict, is the locus of a point in uniform motion along the x axis. Taking this line as a new axis (ict') and in place of x, a new axis x', perpendicular to y, z, and ict', we have a new coördinate system in which our fundamental equation (69) retains complete validity. In other words, as Einstein pointed out, the equations of the electromagnetic field remain true, whatever point is arbitrarily chosen as a point of rest.

---

[24] Abraham-Föppl, II, Equations (30 a) and (30 b).